# The statistical uncertainty associated with the weighted mean frequency in optical frequency comb comparison

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## ABSTRACT

The weighted mean is widely used in combining data sets of experimental measurements with a weight proportional to the value of the data number divided by the sample variance in a conventional method. However, this standard procedure is not appropriate for obtaining the weighted mean frequency of a phase-stabilized signal with white phase noise, since the data are autocorrelated. The autocorrelation is obtained in the case of white phase noise and a new weighting method is proposed. Using this, the uncertainty associated with the weighted mean frequency of a phase-stabilized signal with white phase noise is given. The effect of counter dead-time is also discussed. The result is significantly different from the conventional method, so that the weight is proportional to the square of data number divided by the sample variance, and as a result the uncertainty of the estimated mean is proportional to the sample standard deviation divided by the data number. In an optical frequency comb comparison, frequency combs are phase-locked to a common frequency reference and the frequency difference between optical frequency combs shows white phase noise properties. Thus, it is expected that this new weighting method can be utilized in frequency comparisons of optical frequency combs, settling existing controversy, and providing a way of achieving lower uncertainty.

## FREQUENCY COMPARISON OF OPTICAL FREQUENCY COMBS

Typically, the frequencies of optical frequency combs (OFC) are compared using a common frequency reference. Some examples are shown in Fig. 1 (rf-referenced OFCs (left) [1], and optically referenced OFCs (right) [2]). As the OFCs are phase-locked to a common frequency reference, the frequency difference between OFCs shows white phase noise properties. When the frequency of a signal with white frequency noise is measured, the conventional weighting method, in which the weight is proportional to the value of the data number divided by the sample variance, seems quite reasonable. However, in case of the frequency of a phase-stabilized signal with white phase noise, this conventional weighting method cannot be applied, since the data are autocorrelated. Applying the conventional method to the phase-stabilized case leads to a paradox, so that the data from longer averaging time seem to be over-weighted compared to those from shorter averaging time.

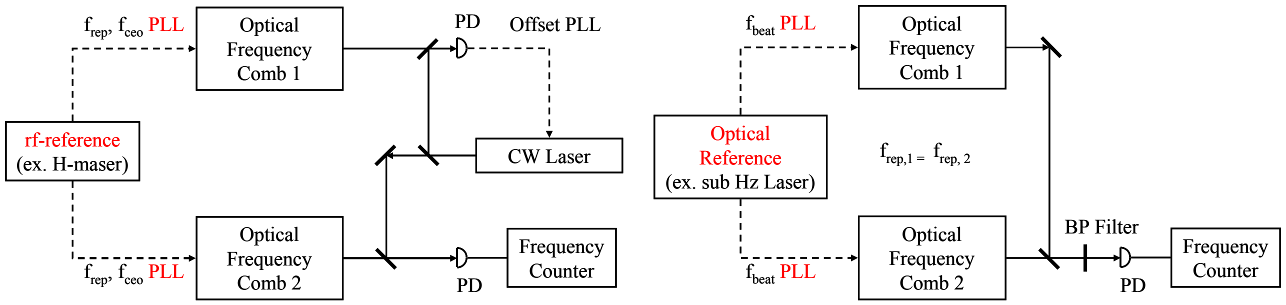


Fig. 1. Frequency Comparison between rf-referenced OFCs (left) and between optically referenced OFCs (right).

## MEASUREMENT WITH A FREQUENCY COUNTER

Let us consider a sinusoidal input signal for a frequency counter given by $V(t) = V_0 \sin[2\pi\nu_0 t + \phi(t)]$, where $V_0$ is the amplitude, $\nu_0$ the carrier frequency, and $\phi(t)$ the phase fluctuation. As a phase-stabilized signal is of interest, it is assumed that $\phi(t)$ has the properties of white phase noise and it is a stationary process with a Gaussian distribution. The instantaneous frequency $\nu$ can be expressed as $\nu = \nu_0 + \Delta\nu(t)$, where the instantaneous frequency deviation from the nominal carrier frequency is defined by $\Delta\nu(t) \equiv d\phi(t)/2\pi dt$.

If the instantaneous frequency $\nu$ is measured by a frequency counter with average time (or gate time) of $\tau$, the k-th sample of the instantaneous frequency deviation, which is defined by

$$\overline{\Delta\nu}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} \Delta\nu(t)dt = \int_{-\infty}^{\infty} \Delta\nu(t) w_\Pi(t-t_k)dt, \tag{1}$$

can be obtained. In the second equality, $w_\Pi(t)$, referred to as a $\Pi$-estimator windowing function following the approach of [3] and [4], is given by

$$w_\Pi(t) = \begin{cases} 1/\tau, & for \quad 0 < t \le \tau \\ 0, & else\,where \end{cases}. \tag{2}$$

This windowing function can be applied in case of a frequency counter, which uses a uniform average over the measurement gate time, including modern high resolution counters, such as Agilent 53132A, in “external arming mode” [3].

The sample variance $S^2_{\overline{\Delta\nu}}(\tau)$ is defined by and can be rewritten as

$$S^2_{\overline{\Delta\nu}}(\tau) \equiv \frac{1}{N-1}\sum_{k=1}^{N}\left(\Delta\nu_k - \mu\right)^2 \approx \left\langle \overline{\Delta\nu}_k{}^2 \right\rangle = \left\langle \left| \int_{-\infty}^{\infty} \Delta\nu(t) w_\Pi(t-t_k)dt \right|^2 \right\rangle, \tag{3}$$

where $\mu$ is the expectation of $\overline{\Delta\nu}$ and can be assumed to be zero for the phase-stabilized signal. This can be easily calculated in Fourier domain. By assuming a low-pass filter with sharp cutoff frequency of $f_h$ ($2\pi f_h \tau >> 1$), the sample variance in the white phase noise case can be expressed as

$$S^2_{\overline{\Delta\nu}}(\tau) = \int_0^{\infty} K f^2 \frac{\sin^2(\pi f \tau)}{(\pi f \tau)^2} df \approx \frac{K f_h}{2\pi^2}\frac{1}{\tau^2}. \tag{4}$$

For a numerical demonstration, data of a phase-stabilized signal (data set #1), which is shown in Fig. 2(a), are acquired. This was obtained by measuring the carrier-envelope offset frequency ($f_{CEO}$) of an optical frequency comb, when the $f_{CEO}$ is phase-stabilized to a stable microwave source. The nominal frequency of the reference microwave was subtracted, so that the data represent the difference between the measured frequency and the reference frequency. A $\Pi$-estimator frequency counter with no dead-time was used with a gate time of 1 s and for a total measurement time of $2^{15}$ s = 32768 s. The original data was processed to produce new data sets for longer gate times. The second data set (data set #2) is produced by averaging the adjacent data pair resulting in a data set with a gate time of 2 s and accordingly with a data number of $2^{14}$ = 16384. Other data sets are produced recursively for longer gate times in a similar way. The produced data are shown in figures 1(b) ~ 1(f) for the gate times of 2 s, 4 s, 8 s, 16 s and 32 s, respectively.

## UNCERTAINTY OF THE WEIGHTED MEAN FREQUENCY WITH WHITE PHASE NOISE

As mentioned in the previous sections, the data of the frequency measurement of a phase-stabilized signal are autocorrelated. There have been some reports on how to calculate the uncertainty of autocorrelated measurements [5]. For the frequency measurement $\left\{\overline{\Delta\nu}_k\right\}$, the weighted mean frequency is given by

$$\overline{\Delta\nu}^w = \frac{\sum_{k=1}^{N}\overline{\Delta\nu}_k}{N}. \tag{5}$$

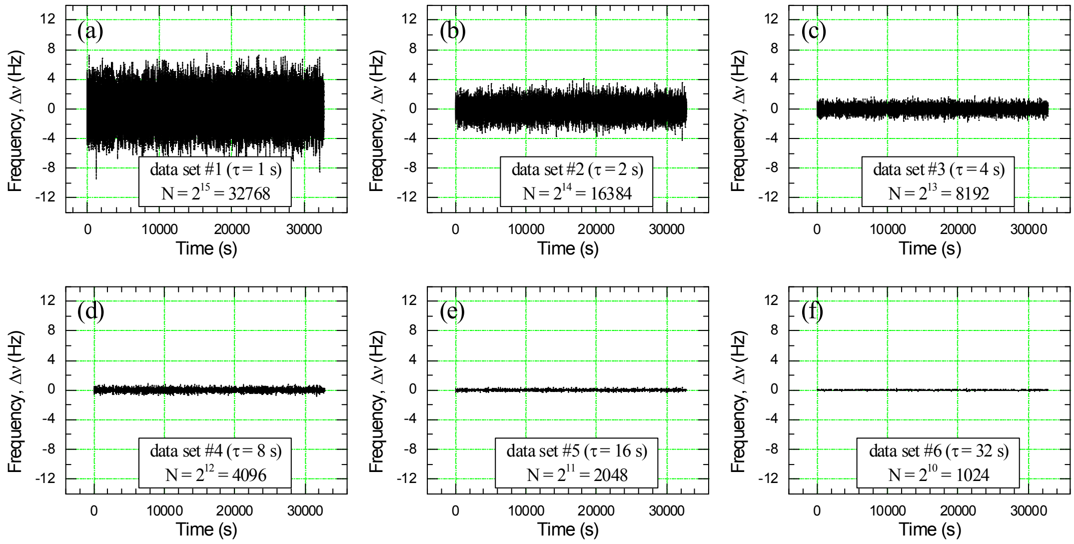


Fig. 2. (a) Numerical example of a frequency measurement of a phase-stabilized signal. A Π-estimator frequency counter with no dead-time was used with a gate time of 1 s and for a total measurement time of $2^{15}$ s = 32768 s. The adjacent data pair were averaged to get the data for the gate times of (b) 2 s, (c) 4 s, (d) 8 s, (e) 16 s and (f) 32 s, respectively.

The uncertainty of this estimated mean frequency is given by

$$u^2_{\overline{\Delta\nu}^w} = \left[1 + \frac{2\sum_{j=1}^{N-1}(N-j)\hat{\rho}(j)}{N}\right]\frac{S^2_{\overline{\Delta\nu}}}{N}, \tag{6}$$

where $S^2_{\overline{\Delta\nu}}$ is the sample variance and $\hat{\rho}(j)$ is the sample autocorrelation at lag $j$, which is given by

$$\hat{\rho}(j) = \frac{\sum_{k=1}^{N-j}\left(\overline{\Delta\nu}_k - \mu\right)\left(\overline{\Delta\nu}_{k+j} - \mu\right)}{\sum_{k=1}^{N}\left(\overline{\Delta\nu}_k - \mu\right)^2} \approx \frac{\left\langle \overline{\Delta\nu}_k\,\overline{\Delta\nu}_{k+j}\right\rangle}{S^2_{\overline{\Delta\nu}}}, \tag{7}$$

where the second equality holds for a phase-stabilized signal, as the expectation of $\overline{\Delta\nu}$ is zero.

## WEIGHTED MEAN BY CONVENTIONAL METHOD

In conventional method, no consideration on the autocorrelation was taken, so that the uncertainty of the estimated mean frequency is given by $S_{\overline{\Delta\nu}}\big/\sqrt{N}$, and the weight is proportional to $N\big/S^2_{\overline{\Delta\nu}}$. The sample variance and the weight for the conventional method for data set #1 ~ #6 were numerically calculated, which are summarized in Table 1. It is expected that each weight of these data sets has the same value, since they were from the same raw data. However, contrary to this expectation, the weight increases as the gate time increases, which is apparently a paradox to be fixed. The main reason for this abnormal tendency is that the sample variance $S^2_{\overline{\Delta\nu}}$ is proportional to $1/\tau^2$ in a white phase noise case as in (4), while the number of data $N$ to $1/\tau$ in these data sets. In spite of this paradox, in a number of research works on the optical frequency comb comparisons, the conventional method has been adopted. However, since the frequency difference between OFCs, which are phase-stabilized to a common (microwave or optical) frequency standard, is measured in these comparisons, the data with longer gate times have been observed to have smaller uncertainties, i.e. higher weight, than expected from those with shorter gate times. It will be shown, in next section, that the conventional method is not appropriate for the weighted mean frequency of the phase-stabilized signal, since these data are not independent but correlated in time domain, and a new method is proposed.

Table 1. The gate time $\tau$, the number of data $N$, the sample standard deviation $S_{\overline{\Delta\nu}}$, and the weight in the conventional method.

| Data Set | $\tau$ (s) | $N$ | $S_{\overline{\Delta\nu}}$ (Hz) | Conventional Weight $\left(N/S^2_{\overline{\Delta\nu}}\right)$ | Relative Weight |
|---|---|---|---|---|---|
| #1 | 1 | 32768 | 2.051 | $7.8\times10^3$ | 1.0 |
| #2 | 2 | 16384 | 1.026 | $1.6\times10^4$ | 2.0 |
| #3 | 4 | 8192 | 0.515 | $3.1\times10^4$ | 4.0 |
| #4 | 8 | 4096 | 0.254 | $6.4\times10^4$ | 8.2 |
| #5 | 16 | 2048 | 0.127 | $1.3\times10^5$ | 16.4 |
| #6 | 32 | 1024 | 0.063 | $2.6\times10^5$ | 33.1 |

## WEIGHTED MEAN BY NEW METHOD

For the signal with white phase noise, it can be shown [6]

$$\hat{\rho}(j)=\frac{\left\langle\overline{\Delta\nu}_{k+j}\overline{\Delta\nu}_k\right\rangle}{S^2_{\overline{\Delta\nu}}}=-\frac{1}{2}\delta(j,1). \tag{8}$$

For a numerical confirmation of these results, the sample autocorrelation function (ACF) for the data introduced in Fig. 2 was calculated numerically. This ACF plot is shown in Fig. 3, which agrees well with the theoretical prediction of equation (8) for the signal with a white phase noise. In Fig. 3, solid lines represent the 95 % confidence band of $\pm1.96/\sqrt{N}$ centered at zero. This confidence band is for the autocorrelations of an uncorrelated sequence. In Fig. 3, $\hat{\rho}(1)\approx-0.5$ and $\hat{\rho}(j)$ is well within the confidence band when $j\geq2$, which implies that there is effectively no correlations.

By inserting the autocorrelation (8) into equation (6), the expression for the uncertainty associated with weighted mean frequency of a phase-stabilized signal is obtained to be

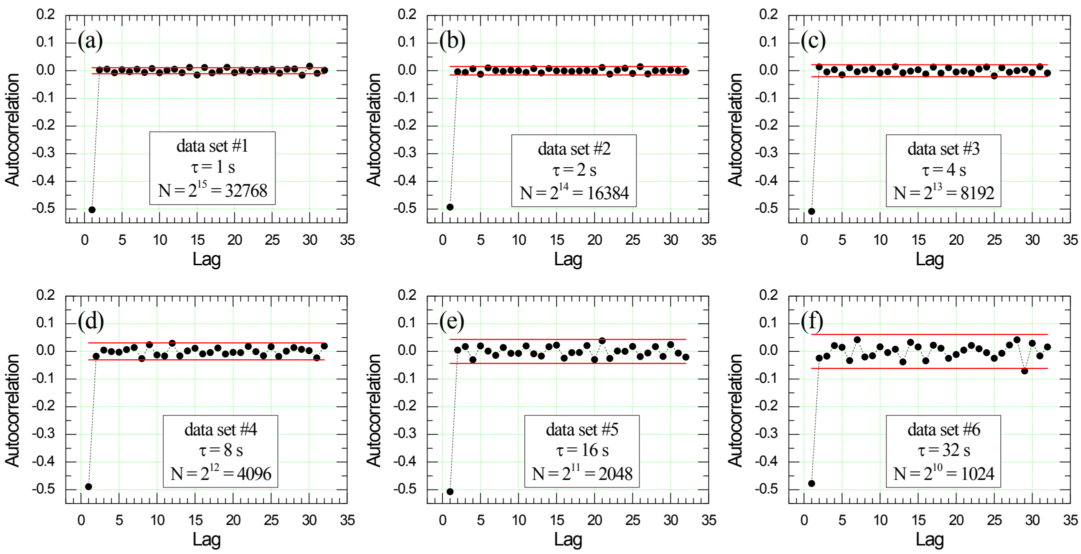


Fig. 3. The sample autocorrelation function (ACF) plot for the data sets introduced in Fig. 2. The solid lines represent the 95 % confidence band of $\pm1.96/\sqrt{N}$ centered at zero.

Table 2. The gate time $\tau$, the number of data $N$, the sample standard deviation $S_{\overline{\Delta\nu}}$, and the weight in the new method.

| Data Set | $\tau$ (s) | $N$ | $S_{\overline{\Delta\nu}}$ (Hz) | New Weight $\left(N^2 / S^2_{\overline{\Delta\nu}}\right)$ | Relative Weight |
|---|---|---|---|---|---|
| #1 | 1 | 32768 | 2.051 | $2.55\times10^8$ | 1.00 |
| #2 | 2 | 16384 | 1.026 | $2.55\times10^8$ | 1.00 |
| #3 | 4 | 8192 | 0.515 | $2.53\times10^8$ | 0.99 |
| #4 | 8 | 4096 | 0.254 | $2.60\times10^8$ | 1.02 |
| #5 | 16 | 2048 | 0.127 | $2.62\times10^8$ | 1.03 |
| #6 | 32 | 1024 | 0.063 | $2.64\times10^8$ | 1.04 |

$$u^2_{\overline{\Delta\nu}^w} = \left[1 + \frac{2(N-1)\left(-\frac{1}{2}\right)}{N}\right]\frac{S^2_{\overline{\Delta\nu}}}{N} = \frac{S^2_{\overline{\Delta\nu}}}{N^2}. \tag{9}$$

Accordingly, the weight for each data set would be proportional to $N^2 / S^2_{\overline{\Delta\nu}}$. As $S^2_{\overline{\Delta\nu}}$ is proportional to $1/\tau^2$ in a white phase noise case and $N^2$ is now also proportional to $1/\tau^2$ in the data sets introduced in section 2.3, the paradox of section 2.3 doesn't exist any more. For a consistency check, the weight for this new method with the data sets #1 ~ #6 was calculated again, the results of which are summarized in Table 2. With this new method, the weights for these data sets have a same value, which is reasonable, since they were from the same raw data. It is noted that the correction for the conventional method originates from the very adjacent data considering the delta function in (8) for the white phase noise case. Thus, when the data sets, that are not adjacent to one another, are combined, the conventional weighting method is still valid.

## EFFECT OF COUNTER DEAD TIME

In case that there exists a dead-time of $\tau_d$ in measuring a frequency of a signal with a white phase noise, the autocorrelation $\hat{\rho}(j)$ is expected to be modified from (8).
It can be shown [6] that in this case

$$\hat{\rho}(j,\tau_d) = -\frac{1}{2}\frac{\sin 2\pi f_h\left[(j-1)\tau + j\tau_d\right]}{2\pi f_h\left[(j-1)\tau + j\tau_d\right]}. \tag{10}$$

$\hat{\rho}(j,\tau_d)$ for $j \geq 2$ can be ignored under the condition of $2\pi f_h \tau >> 1$, while $\hat{\rho}(1,\tau_d)$ is modified to be

$$\hat{\rho}(1,\tau_d) = -\frac{1}{2}\frac{\sin 2\pi f_h \tau_d}{2\pi f_h \tau_d}. \tag{11}$$

It is noted that $\hat{\rho}(1,\tau_d)$ approaches zero, i.e. the correlation disappears, as $\tau_d$ increases, which is shown in Fig. 4. Thus, when $\tau_d$ is prominent, the conventional method of weighting is approximately valid. In an intermediate case, $\hat{\rho}(1,\tau_d)$ can be calculated using the measurement data, and can be inserted in equation (6) to obtain the uncertainty of the weighted mean frequency.

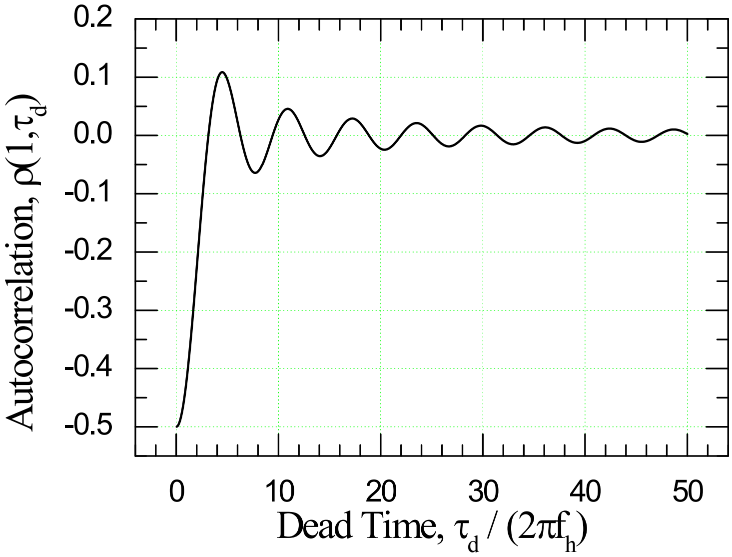


Fig. 4. The autocorrelation $\hat{\rho}(1,\tau_d)$ as a function of the counter dead-time.

**CONCLUSION**

In this paper, an appropriate method for estimating a weighted mean and its uncertainty of frequency measurement data is proposed in case of a phase-stabilized signal with white phase noise. The result is significantly different from the conventional method, so that the weight is proportional to the square of data number divided by the sample variance, and resultantly the uncertainty of the estimated mean is proportional to the sample standard deviation divided by the data number. Especially, it is expected that this new weighting method can be utilized in frequency comparisons of optical frequency combs, settling down existing controversy, and providing a way of achieving lower uncertainty.

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